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Order–chaos transitions in field theories with topological terms: a dynamical systems approach

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Abstract. We present a comparative study of the dynamical behaviour of topological systems of recent interest, namely the non-Abelian Chern–Simons Higgs system and the Yang–Mills Chern–Simons Higgs system. By reducing the full field theories to temporal differential systems by using the assumption of spatially homogeneous fields, we study the Lyapunov exponents for two types of initial conditions. We also examine in minute detail the behaviour of the Lyapunov spectra as a function of the various coupling parameters in the system. We compare our results with those for Abelian Higgs, Yang–Mills Higgs and Yang–Mills Chern–Simons systems which have been discussed recently by other authors. The role of the various terms in the Hamiltonians for such systems in determining the order–disorder transitions is emphasized and shown to be counter-intuitive in the Yang–Mills Chern–Simons Higgs systems.

1. Introduction

Recently, the theory of dynamical systems has provided much insight into the origin of chaos in classical systems which were traditionally thought of as being completely deterministic. However, much of the progress has been mainly in the context of discrete mapping and those differential dynamical systems having low-dimensional phase spaces.

In the context of mathematical physics, differential dynamical systems exist described by a large number of variables and therefore having phase spaces of rather large dimensions. Examples of such systems are the Yang–Mills (YM) system, the Chern–Simons (CS) system and their various enlargements such as the Yang–Mills Higgs (YMH), Yang–Mills Chern–Simons Higgs (YMCSH) and Chern–Simons Higgs (CSH) systems. These systems are treated as dynamical systems after being derived from the original highly nonlinear partial differential equations (PDEs) through the assumption of spatial homogeneity which reduces the dependence of the dynamical variables on the three- or four-dimensional spacetime coordinates to a dependence purely on time. The systems then become temporal differential dynamical systems.

Matinyan *et al* [1] were the first to demonstrate the chaotic nature of gauge theories by treating them as differential dynamical systems. In fact, a detailed investigation has been conducted in this context [2] to classify the dynamical version of the pure YM field theory in terms of its ergodicity properties.

While the condition for ergodic behaviour of a system is that the generic phase trajectory visits all regions of phase space given a sufficiently long time, and all phase averages can

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be replaced by time averages, the YM system exhibits stronger stochasticity properties in phase space. Investigations reveal that it is definitely a mixing system, i.e. no time averaging is required to achieve 'equilibrium'. In contrast to systems which are simply ergodic its spectrum is continuous. Indeed, the studies seem to indicate that the YM dynamical system is a Kolmogorov K-system, which exhibits stronger mixing properties than those mentioned above. A connected neighbourhood of phase trajectories in this case, exhibits a positive average rate of exponential divergence or net positive Lyapunov exponent. Equivalently, by a remarkable theorem [3] a K-system has positive Kolmogorov–Sinai (KS) entropy which is a measure of the degree of chaos, analogous to entropy as a measure of disorder in stastistical mechanics. In fact, while the KS entropy itself is not straightforward to measure, the fact that it is a sum of the positive Lyapunov exponents implies that it is an important concept in the classification of chaotic systems.

Another aspect to the study of gauge theories as dynamical systems is the attempt to understand the ground (vacuum) state structure of quantum chromodynamics (QCD), and also its behaviour in extreme environments (such as high temperature). In this context, systems such as YMH, YMCS and YMCSH have been studied [4].

The addition of the CS term to various Abelian and non-Abelian gauge theories leads to novel features in general, as it is a topological term. Even in the complete PDEs, while a study of the YM case reveals interesting results in connection with the geometry and topology of four-dimensional manifolds, the related CS PDEs have yielded information about three-dimensional manifolds. The symplectic structure of CS theories differs in important ways from that of the Maxwell or YM gauge theories. In perturbative gauge theories the pure CS theories exhibit features that are absent in the Abelian gauge theory with both the Maxwell and CS terms. Delicate aspects relating to the infrared and ultraviolet behaviour and the regularization dependence in such perturbative theories [5], the natural connection of quantized three-dimensional CS gauge theories with two-dimensional conformal field theories [6], and the effect of the YM term which acts as a singular perturbation when added to the $SU(2)$ CS theory [7], have all been extensively explored in the literature.

Another aspect of interest in CS theories relates to its quantum mechanics. Non-perturbative quantum mechanical anomalies in these theories [8], infinite-dimensional symmetry groups that arise in CS quantum mechanics [9], self-dual CS theories and extended supersymmetry [10] are a few areas in which distinct signatures of CS theories are in sharp contrast to those of other gauge theories that have been reported.

In this paper we report on yet another aspect of the CS theories and compare them with other gauge theories. This pertains to the chaotic nature of gauge theories mentioned earlier. For our purpose, the equations of motion are made to evolve only temporally, by suppressing the spatial dependence. While the resulting equations are like the continuum analogues of discrete maps (the latter being an area where extensive work has been done on their chaotic behaviour), they are also the dynamical version of the full gauge theory and hence represent one sector of the corresponding field theory. In the literature this is used as a convenient reduction to examine the integrability properties of the theory. This is because if this sector is proven to be non-integrable the corresponding field theory will also be chaotic [11].

In our recent papers, we have shown that the Abelian CSH system without a kinetic term is integrable, while the inclusion of a kinetic term, making it into a Maxwell Chern–Simons Higgs (MCSH) system (or Yang–Mills with a $U(1)$ symmetry group) yielded a non-integrable system which admitted chaos [12]. The systems were also examined for the Painlevé property. A numerical study of the Lyapunov exponents and phase space trajectories was carried out to show the existence of chaos in these systems. In [13], the analysis was extended to the non-Abelian CSH and YMCSH systems with an $SU(2)$ symmetry group and both were found to be chaotic.

In all of these studies, one aspect that deserves more attention is the possibility of observing the existence of a phase transition, i.e. is there a sharp order to chaos transition in the parameter space of these theories? Of course in these systems energy can also be used as a parameter since it is a conserved quantity. Indeed in [14], it has been claimed that an order–chaos transition is seen within a narrow range of energies. More recently, Kawabe [15, 16] has shown that there are order–chaos transitions for certain ratios of the two parameters in YMH as well as the Abelian Higgs system (YMH with a $U(1)$ symmetry group). Our paper examines this aspect of chaos in YM systems by studying the non-Abelian CSH (NACSH) and the YMCSH systems with an $SU(2)$ symmetry group. A comparative study is done to see the role of the kinetic term, the Higgs term and the CS term in the transition.

Some interesting features regarding the details of the phase transition from order to chaos in the dynamical analogues of both Abelian and non-Abelian gauge theories have been reported in the literature. In the context of Abelian Higgs theories and the YMH theory with a sphaleron solution, Kawabe has reported a transition from order to chaos within a certain range of the Higgs coupling constant and energy [15, 16]. The onset of chaos is remarkably different qualitatively from the corresponding transition in the YMCS system, where Giansanti and Simic [14] have reported the existence of an interesting 'fractal' structure in the phase transition region. Much earlier, Matinyan *et al* [17] observed that the role of the Higgs is to order the YM system and later extensive work on the YMH systems was conducted in [18]. The picture that emerges therefore, is that the Higgs and the CS term have distinct and different roles to play in the transition. It is therefore of importance to examine the effect of both terms on the YM field in the YMCSH theory. As a primer to this, the competing effect of the 'oscillatory' behaviour of the CS term and the 'stabilizing' role of the Higgs term in the CSH system is investigated in this paper. Later, we compare this with the corresponding results in the YMCSH system. It is thus obvious that a K-system such as the YM theory when coupled to CS and Higgs must show a rich and instructive behaviour in the understanding of regularity versus chaos in Hamiltonian systems. A second aspect which emerges is related to an interesting question—will the Higgs stabilize any gauge-invariant term involving only the vector potentials, independent of whether it is of the YM-type or the CS-type or is it necessary that the gauge field is only YM in nature? We attempt to answer these questions in this paper.

2. The dynamical systems

In this section we set up the two systems we shall be examining. Let us consider first the non-Abelian pure CSH system (i.e. without the kinetic term). From our earlier paper [13], the Lagrangian for the non-Abelian ($SU(2)$) CSH (NACSH) system in $2+1$ dimensions in Minkowski space is given by:

$$L = \frac{m}{2}\epsilon^{\mu\nu\lambda}\left[F^a_{\mu\nu}A^a_\alpha - \frac{g}{3}f_{abc}A^a_\mu A^b_\nu A^c_\alpha\right] + D_\mu\phi^\dagger_a D^\mu\phi_a - V(\phi) \tag{1}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc}A^b_\mu A^c_\nu \tag{2}$$

f_{abc} are the structure constants of the $SU(2)$ Lie algebra and

$$D_\mu\phi_a = (\partial_\mu - \mathrm{i}gT^l A^l_\mu)\phi_a. \tag{3}$$

Here, T_a are the generators of the $SU(2)$ algebra, so that $\mathrm{tr}[T_a T_b] = \delta_{ab}$. The equations of motion become:

$$m\epsilon^{\nu\alpha\beta} F^a_{\alpha\beta} = \mathrm{i}g[D^\nu\phi^\dagger T_a\phi - \phi^\dagger T_a D^\nu\phi] \tag{4}$$

$$D_\mu D^\mu(\phi) = -\frac{1}{2}\frac{\partial V}{\partial\phi}. \tag{5}$$

Then, considering the real triplet representation for the Higgs field and the spatially homogeneous solutions $\partial_i A^a_j = \partial_i\phi = 0$ $i, j = 1, 2$ and the gauge choice $A^a_0 = 0$ we get for the $\nu = 0$ component of equation [4]

$$m\boldsymbol{A}_1 \times \boldsymbol{A}_2 = -\boldsymbol{\phi} \times \dot{\boldsymbol{\phi}} \tag{6}$$

which is just the Gauss' law constraint. The remaining equations of motion for the vector field are:

$$\dot{\boldsymbol{A}}_1 = \frac{g^2}{m}(\boldsymbol{A}_2\boldsymbol{\phi}^2 - \boldsymbol{\phi}\boldsymbol{A}_2 \cdot \boldsymbol{\phi}) \tag{7}$$

$$\dot{\boldsymbol{A}}_2 = -\frac{g^2}{m}(\boldsymbol{A}_1\boldsymbol{\phi}^2 - \boldsymbol{\phi}\boldsymbol{A}_1 \cdot \boldsymbol{\phi}). \tag{8}$$

The equation of motion for the Higgs field is:

$$\ddot{\boldsymbol{\phi}} = -g^2[(\boldsymbol{A}_1^2 + \boldsymbol{A}_2^2)\boldsymbol{\phi} - (\boldsymbol{A}_1 \cdot \boldsymbol{\phi}\boldsymbol{A}_1 + \boldsymbol{A}_2 \cdot \boldsymbol{\phi}\boldsymbol{A}_2] - \frac{1}{2}\frac{\partial V}{\partial\boldsymbol{\phi}}. \tag{9}$$

From the equations of motion for the vector fields it is easily seen that $\boldsymbol{A}_1^2 + \boldsymbol{A}_2^2$ is a constant of motion. Throughout this paper, we work with the potential

$$V(\phi) = \frac{\lambda}{4}(\boldsymbol{\phi}^2 - v^2)^2. \tag{10}$$

The NACSH system described by the above equations of motion has three parameters. For comparison with the work of Kawabe [15, 16], we shall scale the variables such that we are left with only one parameter. The following scaling of variables,

$$\begin{aligned}
\boldsymbol{A}_1 &\longrightarrow g\boldsymbol{A}_1\\
\boldsymbol{A}_2 &\longrightarrow g\boldsymbol{A}_2\\
\boldsymbol{\phi} &\longrightarrow \frac{g}{\sqrt{m}}\boldsymbol{\phi}\\
v &\longrightarrow \frac{g}{\sqrt{m}}v
\end{aligned}$$

reduces the equations of motion to:

$$\dot{\boldsymbol{A}}_1 = [\boldsymbol{A}_2\boldsymbol{\phi}^2 - \phi(\boldsymbol{A}_2 \cdot \phi)] \tag{11}$$

$$\dot{\boldsymbol{A}}_2 = -[\boldsymbol{A}_1\boldsymbol{\phi}^2 - \phi(\boldsymbol{A}_1 \cdot \phi)] \tag{12}$$

$$\ddot{\boldsymbol{\phi}} = -[(\boldsymbol{A}_1^2 + \boldsymbol{A}_2^2)\boldsymbol{\phi} - (\boldsymbol{A}_1 \cdot \boldsymbol{\phi}\boldsymbol{A}_1 + \boldsymbol{A}_2 \cdot \boldsymbol{\phi}\boldsymbol{A}_2)] - \frac{\kappa}{2}\phi(\boldsymbol{\phi}^2 - v^2) \tag{13}$$

with $\kappa = \lambda m/g^2$. In the rest of the paper, we shall set the scaled v to be one without loss of generality. The scaled Lagrangian is given by

$$L = (\dot{\boldsymbol{A}}_1 \cdot \boldsymbol{A}_2 - \dot{\boldsymbol{A}}_2 \cdot \boldsymbol{A}_1) + \dot{\boldsymbol{\phi}}^2 - [(\boldsymbol{A}_1^2 + \boldsymbol{A}_2^2)\boldsymbol{\phi}^2 - (\boldsymbol{A}_1 \cdot \phi)^2 - (\boldsymbol{A}_2 \cdot \phi)^2] - \frac{\kappa}{4}(\boldsymbol{\phi}^2 - 1)^2. \tag{14}$$

The corresponding energy function is easily seen to be

$$E = \dot{\boldsymbol{\phi}}^2 + [(\boldsymbol{A}_1^2 + \boldsymbol{A}_2^2)\boldsymbol{\phi}^2 - (\boldsymbol{A}_1 \cdot \phi)^2 - (\boldsymbol{A}_2 \cdot \phi)^2] + \frac{\kappa}{4}(\boldsymbol{\phi}^2 - 1)^2. \tag{15}$$

These are the equations governing the NACSH dynamical system. Since we wish to compare the results for the NACSH system with those of the YMCSH system we proceed to set up the YMCSH dynamical equations. The Lagrangian is given by

$$L = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{m}{2}\epsilon^{\mu\nu\alpha}\left[F^a_{\mu\nu}A^a_\alpha - \frac{g}{3}f_{abc}A^a_\mu A^b_\nu A^c_\alpha\right] + D_\mu\phi^\dagger_a D^\mu\phi_a - V(\phi). \tag{16}$$

The equations of motion are

$$D_\mu F^{\mu\nu a} + m\epsilon^{\nu\alpha\beta}F^a_{\alpha\beta} = \mathrm{i}g[D^\nu\phi^\dagger T_a\phi - \phi^\dagger T_a D^\nu\phi] \tag{17}$$

$$D_\mu D^\mu\phi^a = -\frac{\partial V}{\partial\phi^*_a} = -\frac{1}{2}\frac{\partial V}{\partial\phi_a}. \tag{18}$$

The $\nu = 0$ component gives the Gauss' law constraint which in this case is

$$\tfrac{1}{2}(\boldsymbol{A}_1 \times \dot{\boldsymbol{A}}_1 + \boldsymbol{A}_2 \times \dot{\boldsymbol{A}}_2 + 2m\boldsymbol{A}_1 \times \boldsymbol{A}_2) = -2\boldsymbol{\phi} \times \dot{\boldsymbol{\phi}}. \tag{19}$$

Once again, we choose the gauge $A_0 = 0$ and consider the spatially homogeneous case; then the equations of motion for the gauge fields become:

$$\ddot{\boldsymbol{A}}_1 + 2m\dot{\boldsymbol{A}}_2 + 2g^2(\boldsymbol{A}_1\boldsymbol{\phi}^2 - \boldsymbol{\phi}\boldsymbol{A}_1\cdot\boldsymbol{\phi}) + g^2(\boldsymbol{A}_1\boldsymbol{A}_2\cdot\boldsymbol{A}_2 - \boldsymbol{A}_2\boldsymbol{A}_1\cdot\boldsymbol{A}_2) = 0 \tag{20}$$

$$\ddot{\boldsymbol{A}}_2 - 2m\dot{\boldsymbol{A}}_1 + 2g^2(\boldsymbol{A}_2\boldsymbol{\phi}^2 - \boldsymbol{\phi}\boldsymbol{A}_2\cdot\boldsymbol{\phi}) + g^2(\boldsymbol{A}_2\boldsymbol{A}_1\cdot\boldsymbol{A}_1 - \boldsymbol{A}_1\boldsymbol{A}_1\cdot\boldsymbol{A}_2) = 0. \tag{21}$$

From these equations it is easy to see that

$$\boldsymbol{A}_2\cdot\dot{\boldsymbol{A}}_1 - \boldsymbol{A}_1\cdot\dot{\boldsymbol{A}}_2 + m(\boldsymbol{A}_1^2 + \boldsymbol{A}_2^2)$$

is a constant of motion. The equation of motion for the three component Higgs field becomes

$$\ddot{\boldsymbol{\phi}} = -g^2[(\boldsymbol{A}_1^2 + \boldsymbol{A}_2^2)\boldsymbol{\phi} - (\boldsymbol{A}_1\cdot\boldsymbol{\phi}\boldsymbol{A}_1 + \boldsymbol{A}_2\cdot\boldsymbol{\phi}\boldsymbol{A}_2)] - \frac{1}{2}\frac{\partial V}{\partial\boldsymbol{\phi}}. \tag{22}$$

In this case we have three second-order differential equations for each vector field. The Lagrangian which leads to these equations of motion is:

$$\begin{aligned} L = {} & \tfrac{1}{2}(\dot{\boldsymbol{A}}_1^2 + \dot{\boldsymbol{A}}_2^2) + m(\dot{\boldsymbol{A}}_1\cdot\boldsymbol{A}_2 - \dot{\boldsymbol{A}}_2\cdot\boldsymbol{A}_1) + \dot{\boldsymbol{\phi}}^2 \\ & -g^2[\tfrac{1}{2}(\boldsymbol{A}_1^2\boldsymbol{A}_2^2 - (\boldsymbol{A}_1\cdot\boldsymbol{A}_2)^2) + (\boldsymbol{A}_1^2 + \boldsymbol{A}_2^2)\boldsymbol{\phi}^2 - (\boldsymbol{A}_1\cdot\boldsymbol{\phi})^2 - (\boldsymbol{A}_2\cdot\boldsymbol{\phi})^2] \\ & -V(\phi). \end{aligned} \tag{23}$$

Using the same rescaling as for NACSH we have the equations of motion

$$\frac{1}{m}\ddot{\boldsymbol{A}}_1 + 2\dot{\boldsymbol{A}}_2 + 2(\boldsymbol{A}_1\boldsymbol{\phi}^2 - \boldsymbol{\phi}\boldsymbol{A}_1\cdot\boldsymbol{\phi}) + \frac{1}{m}(\boldsymbol{A}_1\boldsymbol{A}_2\cdot\boldsymbol{A}_2 - \boldsymbol{A}_2\boldsymbol{A}_1\cdot\boldsymbol{A}_2) = 0 \tag{24}$$

$$\frac{1}{m}\ddot{\boldsymbol{A}}_2 - 2\dot{\boldsymbol{A}}_1 + 2(\boldsymbol{A}_2\boldsymbol{\phi}^2 - \boldsymbol{\phi}\boldsymbol{A}_2\cdot\boldsymbol{\phi}) + \frac{1}{m}(\boldsymbol{A}_2\boldsymbol{A}_1\cdot\boldsymbol{A}_1 - \boldsymbol{A}_1\boldsymbol{A}_1\cdot\boldsymbol{A}_2) = 0 \tag{25}$$

and

$$\ddot{\boldsymbol{\phi}} = -[(\boldsymbol{A}_1^2 + \boldsymbol{A}_2^2)\boldsymbol{\phi} - (\boldsymbol{A}_1\cdot\boldsymbol{\phi}\boldsymbol{A}_1 + \boldsymbol{A}_2\cdot\boldsymbol{\phi}\boldsymbol{A}_2] - \frac{\kappa}{2}\boldsymbol{\phi}(\boldsymbol{\phi}^2 - 1). \tag{26}$$

It is interesting to note that while in the NACSH system the YM parameter g, the Higgs parameter λ and the CS parameter m could all be combined into the parameter κ, this is not possible for the YMCSH system where we are left with both κ and m appearing explicitly. The scaled Lagrangian in this case is given by

$$\begin{aligned} L = {} & \frac{1}{2m}(\dot{\boldsymbol{A}}_1^2 + \dot{\boldsymbol{A}}_2^2) + (\dot{\boldsymbol{A}}_1\cdot\boldsymbol{A}_2 - \dot{\boldsymbol{A}}_2\cdot\boldsymbol{A}_1) + \dot{\boldsymbol{\phi}}^2 - \frac{1}{2m}[\boldsymbol{A}_1^2\boldsymbol{A}_2^2 - (\boldsymbol{A}_1\cdot\boldsymbol{A}_2)^2] \\ & +[(\boldsymbol{A}_1^2 + \boldsymbol{A}_2^2)\boldsymbol{\phi}^2 - (\boldsymbol{A}_1\cdot\boldsymbol{\phi})^2 - (\boldsymbol{A}_2\cdot\boldsymbol{\phi})^2] - \frac{\kappa}{4}(\boldsymbol{\phi}^2 - 1)^2 \end{aligned} \tag{27}$$

with an energy function

$$E = \frac{1}{2m}(\dot{\boldsymbol{A}}_1^2 + \dot{\boldsymbol{A}}_2^2) + \dot{\phi}^2 + \frac{1}{2m}[\boldsymbol{A}_1^2\boldsymbol{A}_2^2 - (\boldsymbol{A}_1\cdot\boldsymbol{A}_2)^2]$$
$$+[(\boldsymbol{A}_1^2 + \boldsymbol{A}_2^2)\phi^2 - (\boldsymbol{A}_1\cdot\phi)^2 - (\boldsymbol{A}_2\cdot\phi)^2] + \frac{\kappa}{4}(\phi^2-1)^2. \tag{28}$$

This completes the description of the dynamical systems which we shall be studying. In the next section, we describe the numerical analysis that we have carried out.

3. Numerical analysis

It is clear from the above dynamical equations that the phase spaces of these systems are large. One of the traditional ways for examining phase spaces to determine chaotic versus regular behaviour has been to use the Poincaré sections where one examines the points mapped out on a plane surface in phase space as the trajectory crosses it. While this is straightforward for dynamical systems whose phase-space dimensionality does not exceed four, it is difficult to interpret it in the systems we are dealing with. We therefore examine other signatures of chaos such as the Lyapunov exponent, phase trajectories etc.

3.1. The NACSH system

We have examined the variation of the maximal Lyapunov exponent as the two NACSH parameters E and κ are varied. This clearly shows us regions of regular behaviour (where the exponent goes to zero) and regions of chaotic behaviour (where the exponent is positive). These calculations were carried out for a wide range of initial conditions. The initial conditions that were chosen were in turn dictated by the dynamical systems themselves. Being derived from the equations of motion the field variables are required to satisfy the Gauss' law constraint. Since this constraint must be preserved during the time evolution of the dynamical system, it is sufficient to ensure their validity via the initial conditions.

For the NACSH system the following forms were chosen as the initial conditions:

$$\boldsymbol{A}_1 = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{A}_2 = \begin{pmatrix} 0 \\ x \\ 0 \end{pmatrix} \qquad \phi = \begin{pmatrix} -x \\ x \\ 0 \end{pmatrix} \qquad \dot{\phi} = \frac{1}{2}\begin{pmatrix} x \\ x \\ 0 \end{pmatrix}. \tag{29}$$

For YMCSH this is supplemented with

$$\dot{\boldsymbol{A}}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \dot{\boldsymbol{A}}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{30}$$

x was then varied to obtain a range of initial conditions for suitable energies of interest. In figures 1–4 we show the behaviour of the maximal Lyapunov exponent as a function of energy for $\kappa = 0, 0.5, 1, 5$. For $\kappa = 0$ (i.e. absence of the Higgs potential) we see that the system is mostly *chaotic* with a window of regularity for $7 \leqslant E \leqslant 9$. By increasing the value of κ, we find a transitional region of regular to chaotic behaviour at small energies, in contrast to mostly oscillatory behaviour at higher energies. For $\kappa = 1$, more transitions from order to chaos appear at larger energies. Much more dramatic behaviour is seen for large κ ($\kappa = 5$), where oscillatory behaviour manifests itself for $E \geqslant 3$. Thus, we see that the effect of the topological term (large m) is to produce a regular oscillatory behaviour in the dynamics of the system.

To illustrate the regular and chaotic behaviour on the trajectories of this system we exhibit in figures 5 and 6 phase plots corresponding to the energies at which the

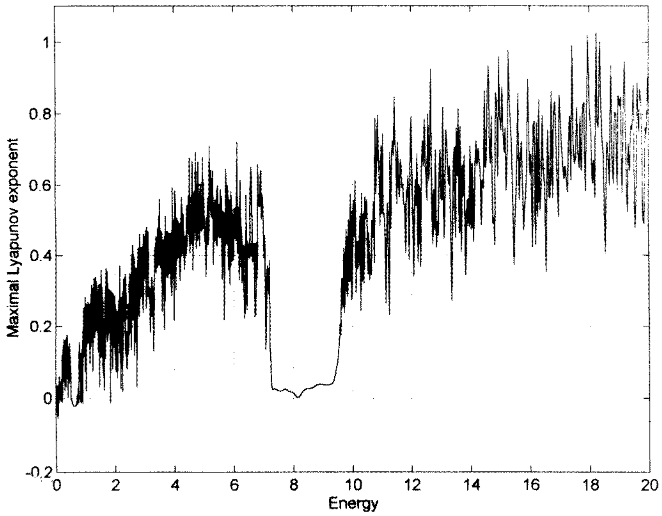

Figure 1. Variation of the maximal Lyapunov exponent with energy for NACSH for $\kappa = 0$.

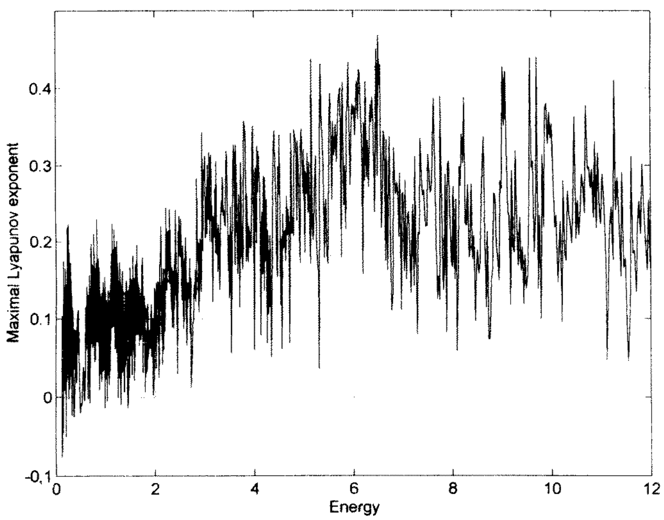

Figure 2. Variation of the maximal Lyapunov exponent with energy for NACSH for $\kappa = 0.5$.

Lyapunov exponent shows regular and chaotic behaviour. These correspond to $\kappa = 1$ and $E = 10.28, 20.607$.

Giansanti and Simic [14] report a ‘fractal’-like structure of order–chaos transitions in YMCS systems. In our particular case, when the YM field is absent and the Higgs field is present we see no such ‘fractal’ behaviour in the region of phase space that we have examined. Therefore, this suggests that the quartic coupling that arises from the inclusion of the kinetic YM term may be responsible for the observed ‘fractal’ structure. This rich phase space structure of the CSH, YMCS and YMCSH systems clearly needs further exploration. The set of trajectories examined by Giansanti and Simic do not correspond to the ansatz that we have chosen and therefore we have explored different regions of phase space. Thus, our results are complementary to those obtained by Giansanti and Simic [14].

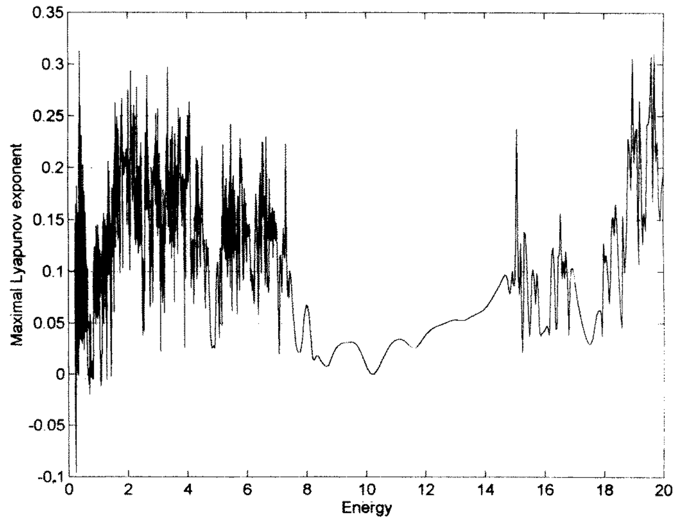

Figure 3. Variation of the maximal Lyapunov exponent with energy for NACSH for $\kappa = 1.0$.

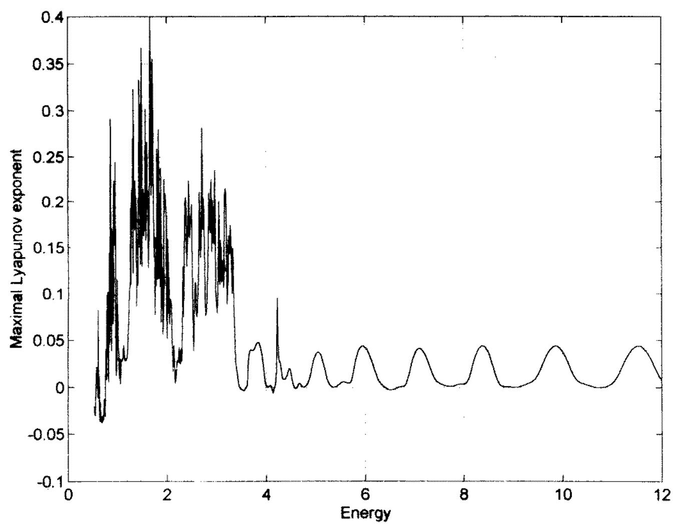

Figure 4. Variation of the maximal Lyapunov exponent with energy for NACSH for $\kappa = 5.0$.

We have examined the NACSH system for another ansatz which we may call the two-variable ansatz:

$$\boldsymbol{A}_1 = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{A}_2 = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} \qquad \boldsymbol{\phi} = \begin{pmatrix} -x \\ y \\ 0 \end{pmatrix} \qquad \dot{\boldsymbol{\phi}} = \frac{1}{2}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}. \tag{31}$$

For YMCSH this is supplemented with

$$\dot{\boldsymbol{A}}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \dot{\boldsymbol{A}}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{32}$$

This allows us to compare our results with those of Kawabe for the Abelian Higgs theory

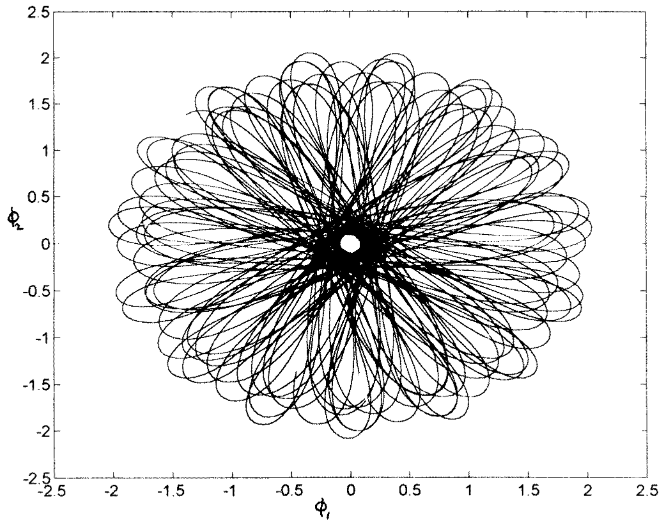

Figure 5. Phase plot of ϕ_2 versus ϕ_1 showing regular behaviour for NACSH for the one-variable ansatz for $\kappa = 1$, $E = 10.2088$ and $x = 1.381$.

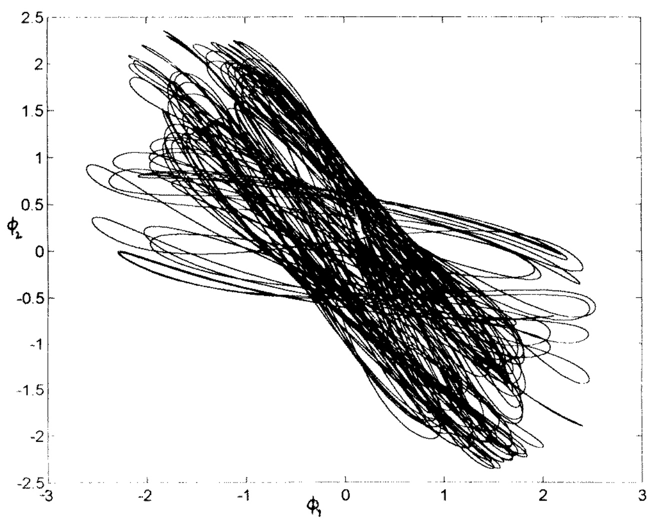

Figure 6. Phase plot of ϕ_2 versus ϕ_1 showing chaotic behaviour for NACSH for the one-variable ansatz for $\kappa = 1$, $E = 20.6071$ and $x = 1.64$.

[15], where motion is completely bounded for $Q = 4Eg^2/\lambda \leqslant 1$, while it is unbounded for $Q > 1$. In contrast, for the CSH system that we are studying, for all values of $Q = 4E/\kappa$ and $\kappa = \lambda/mg^2$, the motion is bounded. Indeed we find that for large E and large κ the contour levels of the function $W = [(\boldsymbol{A}_1^2 + \boldsymbol{A}_2^2)\phi^2 - (\boldsymbol{A}_1 \cdot \phi)^2 - (\boldsymbol{A}_2 \cdot \phi)^2] + \frac{\kappa}{4}(\phi^2 - 1)^2$ show extremely restricted domains for the dynamics. Figure 7 represents the potential contours for $\kappa = 10$ for energies in the range $\{0.25, 2\}$ while figure 8 gives the contours for the energy range $\{2, 10\}$. There is a dramatic change in the available phase space for this particular two-variable ansatz. Moreover, it should be remembered that $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are not independent degrees of freedom in the model. They are actually canonical conjugates

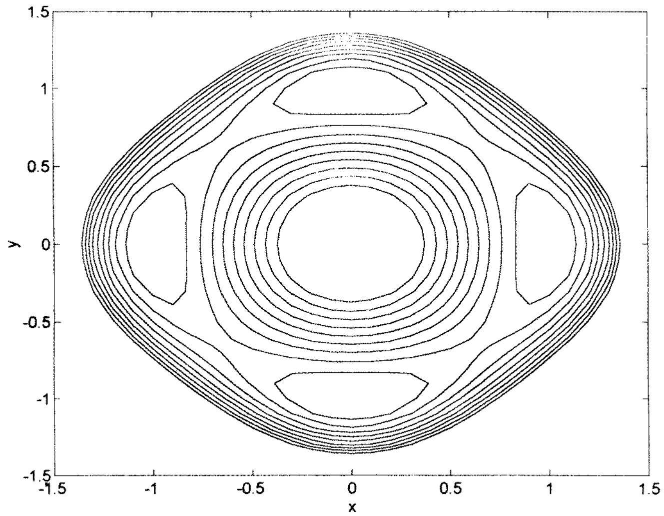

Figure 7. Contour plot for the potential corresponding to NACSH for $\kappa = 10$ and energy range $\{0.25, 2\}$.

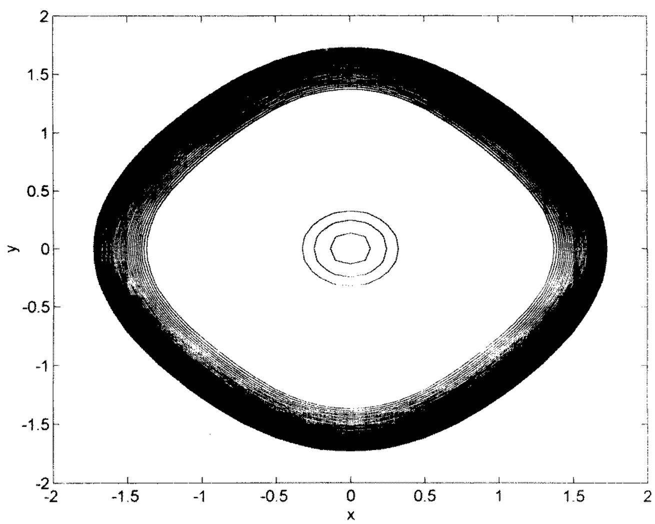

Figure 8. Contour plot for the potential corresponding to NACSH for $\kappa = 10$ and energy range $\{2, 10\}$.

of each other and it is the equations of motion in equations (7)–(9) which reveal the full complexity of the problem. These clearly indicate that the non-Abelian nature of the CS term is already contributing to a significant change in the dynamical structure.

Evidence for the transition from regularity to chaos is seen in figure 9, where we show the fraction of the phase space that is regular. This is obtained by calculating the maximal Lyapunov exponent for $\kappa = 1$ for various initial conditions for energies ranging from 1 to 10. The cut-off on the exponent for regularity was taken to be 0.01. A simple count on the exponents falling below this value out of 100 initial conditions for each energy was carried out. The figure clearly reveals that for small κ, the NACSH system becomes chaotic more

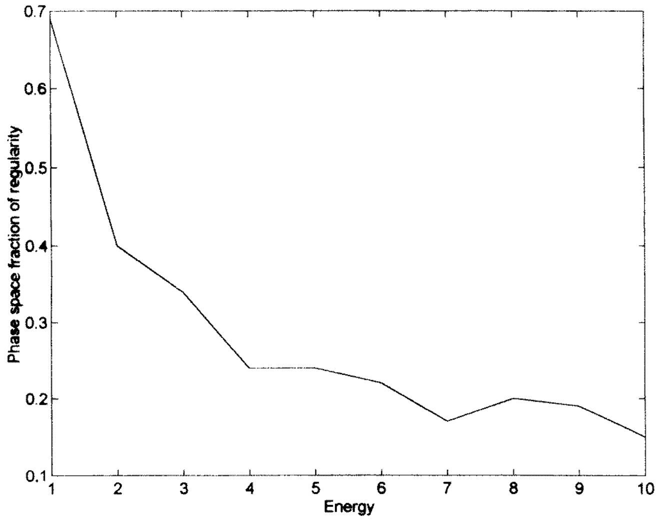

Figure 9. Phase space fraction of regularity versus energy for $\kappa = 1$ for NACSH.

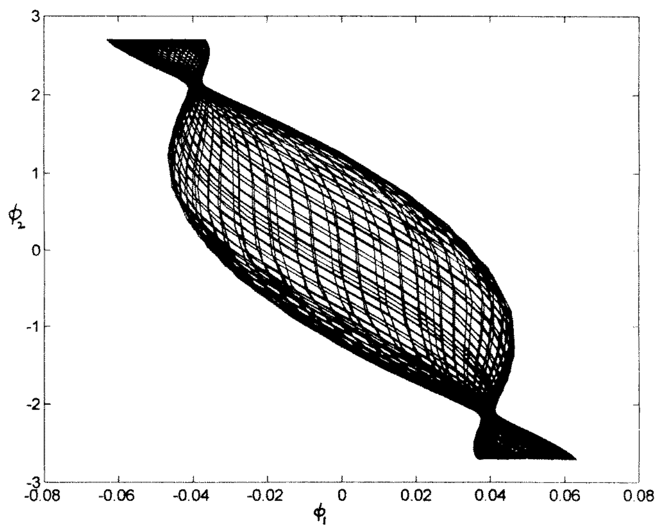

Figure 10. Phase plot of ϕ_2 versus ϕ_1 showing regular behaviour for NACSH for the two-variable ansatz for $\kappa = 1$, $E = 10$, $x = 0.05$ and $y = 2.598\,405$

or less monotonically as the energy is increased. However, for large κ there is no such simple behaviour. This is discussed further in the next section.

This transition between regular and chaotic behaviour is also seen in the phase space trajectories shown in figures 10 and 11. In figure 10 we plot ϕ_2 versus ϕ_1 for $\kappa = 1$, $E = 10$ and an initial condition where the maximal Lyapunov exponent is almost zero. In figure 11 we show the phase plot for the same parameter values for an initial condition which gives a large maximal Lyapunov exponent. As expected, the trajectory in figure 10 corresponding to a very small value of the Lyapunov exponent is regular, whereas the one in figure 11 corresponding to a large value is chaotic. The regular trajectory is not associated with any

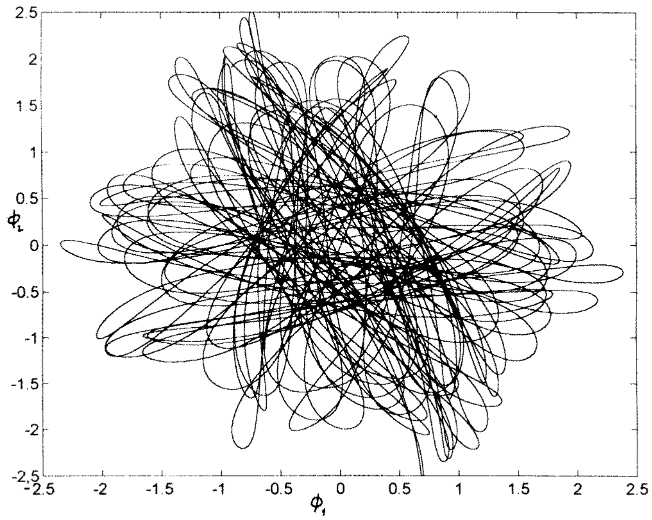

Figure 11. Phase plot of ϕ_2 versus ϕ_1 showing chaotic behaviour for NACSH for the two-variable ansatz for $\kappa = 1$, $E = 10$, $x = 0.9$ and $y = 1.908\,953$.

additional first integral of motion and its nature is due to the initial condition. For this initial condition, ϕ_1 and ϕ_2 appear to decouple and have independent oscillatory motions.

3.2. The YMCSH system

We now proceed to the numerical investigation of the YMCSH system. We have extensively studied the variation of the maximal Lyapunov exponent as a function of energy in the range 1–100, for values of κ ranging from 1 to 200, for the one-variable ansatz. For small κ which corresponds to a small value of the Higgs coupling constant or the CS term or a large value of the gauge coupling constant, the system is almost completely chaotic, as is expected. Even for κ as large as 10, the system exhibits more regularity only at lower energies. This is illustrated in figure 12 which shows the variation of the maximal Lyapunov exponent as a function of energy for $\kappa = 1, 10$ with $m = 1$. For larger values of κ we observe some very interesting effects. For $\kappa = 20$ we observe 'violent' order–chaos transitions for energies as large as $E = 75$, as illustrated in figure 13, which cannot be explained by any simple scaling argument. The Lyapunov exponent, expressed as a function of energy for $\kappa = 50$, 100 and 200, is displayed in figures 14(a)–(c). Here we notice an approximate scaling behaviour. For $E \leqslant 0.4\kappa$, the system has large regions of regularity characterized by a small value of the exponent, interspersed with bands of chaos with a remarkable similarity in structure.

Different interesting results on gauge-Higgs systems have been previously reported in the literature. Matinyan *et al* [17] have considered a simplified version of the YMH system by freezing the Higgs field at its vacuum expectation value and retaining only two degrees of freedom for the gauge fields. In this case, when the energy is less than a known function of the parameters of the model the system is nearly integrable, but it is chaotic when the energy is greater. The YMH system, with a monopole solution considered by Kawabe [18], is regular beyond a certain value of κ for all values of the energy. Obviously our model with a different dynamics displays a different behaviour. The existence of regularity for values of E/κ below some critical value has been reported for the Abelian Higgs model [15], as well as the YMH system for sphaleron-like solutions [16]. The difference here is

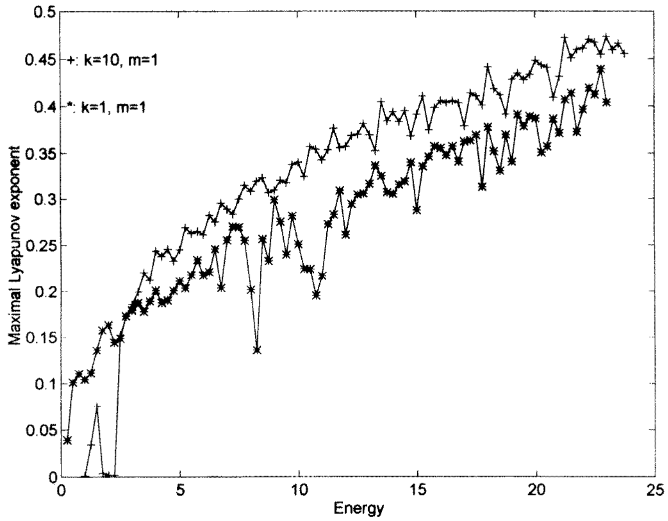

Figure 12. Comparison of maximal Lyapunov exponent versus energy for YMCSH for the one-variable ansatz for $\kappa = 1$, $m = 1$ ($*$ represents the curve) with $\kappa = 10$, $m = 1$ ($+$ represents the curve).

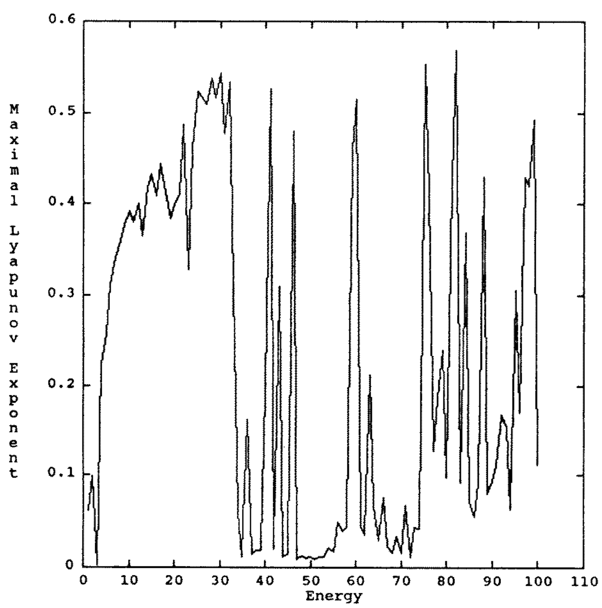

Figure 13. Variation of the maximal Lyapunov exponent with energy for YMCSH for $\kappa = 20$, for the one-variable ansatz.

that the regular region has a great deal of structure with order–chaos transitions, obviously due to the CS term. It is very significant that even for very high values of κ, a non-trivial behaviour is observed at low energies. We have found that even when κ has as large a value as 200, the Higgs field is not necessarily pushed to the Higgs sphere $\phi^2 = 1$. In fact, it is interesting that for some initial conditions, the Lyapunov exponent is large even if

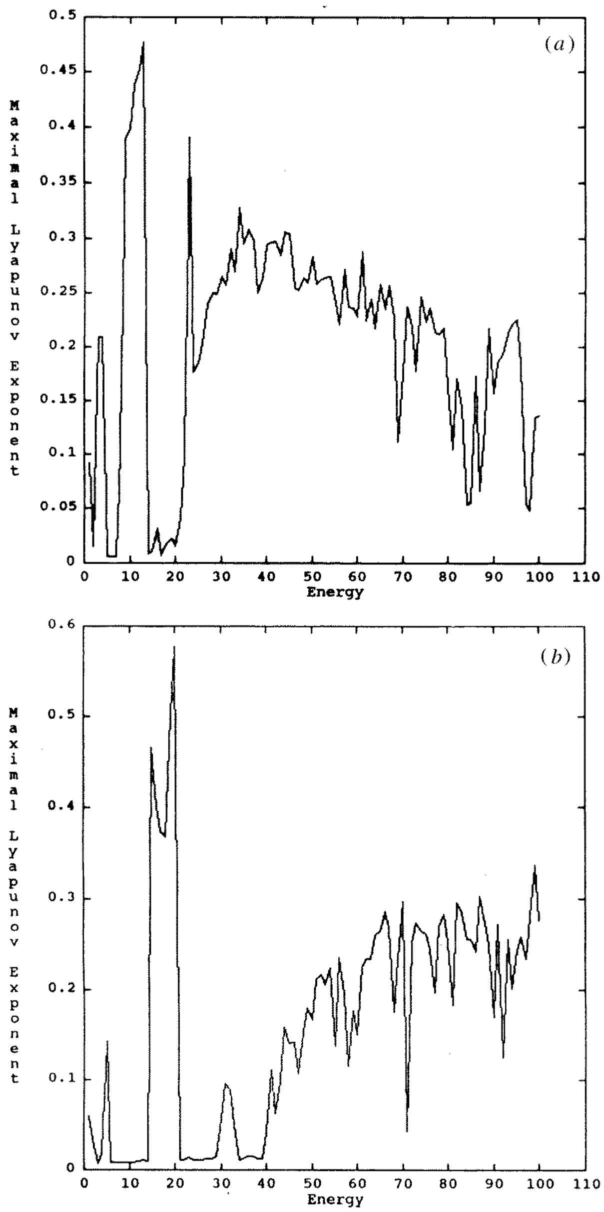

Figure 14. Variation of the maximal Lyapunov exponent with energy for YMCSH for (*a*) $\kappa = 50$, (*b*) $\kappa = 100$ and (*c*) $\kappa = 200$, for the one-variable ansatz.

the Higgs fields are driven close to the Higgs sphere, and for some other initial conditions, the exponent is almost zero even when they are far away from the sphere. Clearly, a complicated interplay of the Higgs potential, the YM term and the CS term is responsible for the rich structure that we obtain. In fact, for the same value of energy and κ also the system displays a varied behaviour. Figure 15 shows the maximal Lyapunov exponent as a function of the initial variable x for the two-variable ansatz, for fixed values of energy and κ. Once again we see regions of regularity for small energies and chaotic behaviour for

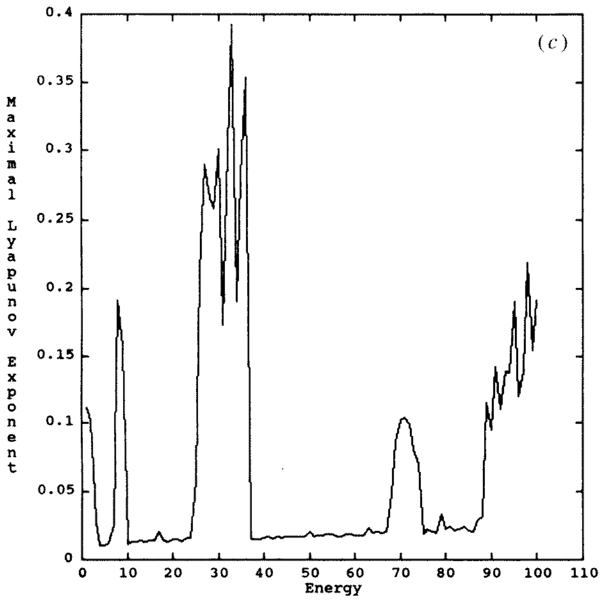

Figure 14. (Continued)

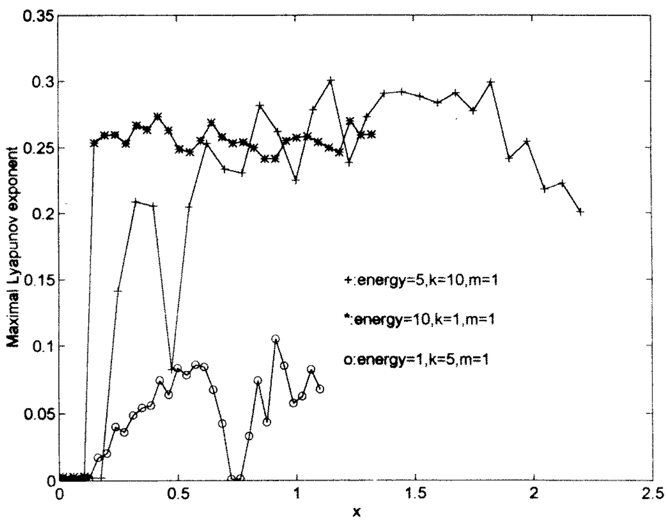

Figure 15. Comparison of maximal Lyapunov exponent versus x for YMCSH for the two-variable ansatz for $\kappa = 5$, $E = 1$, $m = 1$ (○ represents the curve) with $\kappa = 10$, $E = 5$, $m = 1$ (+ represents the curve) and $\kappa = 1$, $E = 10$, $m = 1$ ($*$ represents the curve).

large energies irrespective of the value of κ. Further evidence for regularity and chaos for the two-variable ansatz is given in figures 16 and 17. In figure 16 we see that for $\kappa = 5$ and $E = 1$, the phase space trajectory is highly regular and quasiperiodic. Figure 17 shows a region for the same parameter values, where the phase space is chaotic.

All the calculations for the NACSH system were carried out by using a straightforward Runge–Kutta fourth-order routine with care being taken to preserve the constants of motion

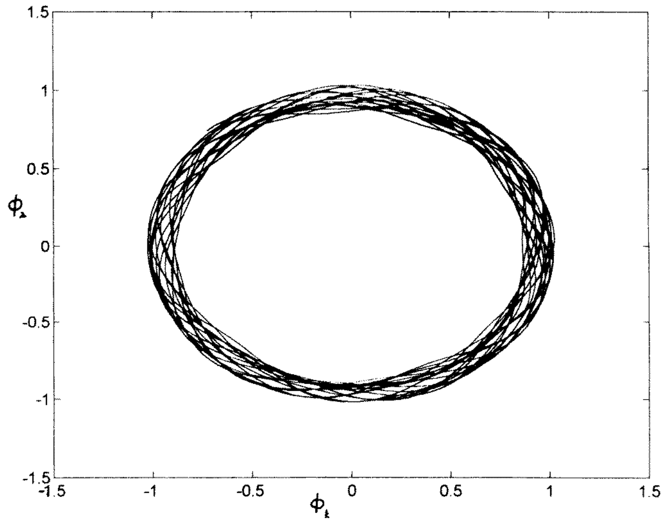

Figure 16. Phase plot of ϕ_2 versus ϕ_1 showing regular behaviour for YMCSH for the two-variable ansatz for $\kappa = 5$, $E = 1$, $x = 0.725$ and $y = 0.742\,085$.

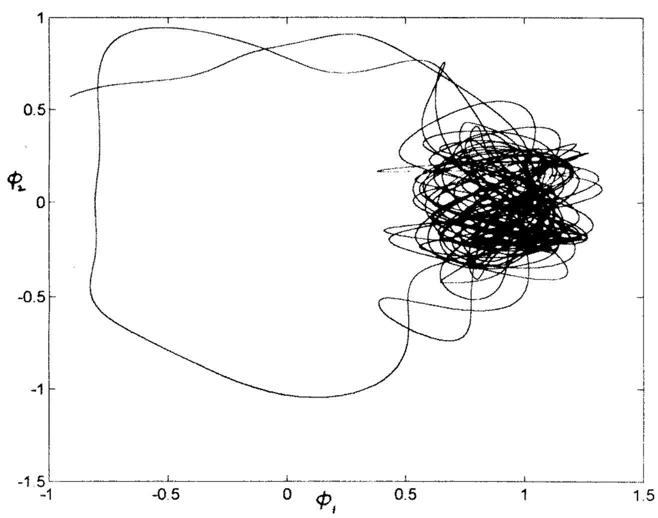

Figure 17. Phase plot of ϕ_2 versus ϕ_1 showing chaotic behaviour for YMCSH for the two-variable ansatz for $\kappa = 5$, $E = 1$, $x = 0.9125$ and $y = 0.571\,046$.

to an accuracy of one part in 10^5. However, the calculations for the YMCSH system required an adaptive step-size Runge–Kutta routine to ensure energy conservation to the same degree of accuracy as in the NACSH system.

4. Results and discussion

From the numerical studies undertaken by us, certain very interesting features emerge, not only regarding the richness of the phase space corresponding to gauge theories, but also with respect to some 'counter-intuitive' phenomena that occur in these systems.

One feature that seems to be common to both the YMH [17] and CSH systems, from an investigation of the potential contours corresponding to various initial conditions, is the boundedness of phase space. This is in contrast to the situation that prevails in the Abelian Higgs system [15]. The boundedness observed is understood better when we realize that the YMCS dynamical system can be used to describe particle motion in a magnetic field [14], a physical situation which allows only for bounded motion.

Apart from the crucial role played by the ansatz chosen by us, in this matter, the non-Abelian nature of the gauge term also plays an important part. In this context, we realize that the boundedness observed is quite independent of the details of the non-Abelian coupling, i.e. whether it is of the YM-type or of the trilinear CS-type.

We recall that the YM system is a K-system [2] and that the CS term creates more regular windows when added to the YM system [14]. This accounts to a large measure, for the fact that for the same κ and m values, the maximal Lyapunov exponents in the YMCSH case are larger than the corresponding ones for the CSH system.

Most of the work reported in the literature on YM, YMCS and YMH dynamical systems deal with reduced numbers of degrees of freedom using simple ansatzes. We have studied the CSH and YMCSH systems in detail retaining all the essential degrees of freedom over a large range of values of κ and energy. The oscillatory effect of the CS term leading to order–chaos transitions observed earlier for small values of energy has been shown to exist for larger energies. Similarly, the regularizing role of the Higgs term for values of E/κ smaller than some value is very much present in the YMCSH system. However, some features emerge, which are in contrast to what one would expect from naive arguments. For instance, the CS term produces strong oscillatory effects even in 'regions of chaos' (see figure 13). More striking is the fact that even for as large a value of κ as 200, there are large bands of chaos in the 'regular regions', i.e. for $E \leqslant 0.4\kappa$. This is counter-intuitive in the sense that one would expect the Higgs self-coupling term to dominate the dynamics for large κ and move ϕ^2 towards the Higgs sphere. Thus, the interplay between the oscillatory effect of the CS term, the regularizing effect of the Higgs term and the completely chaotic nature of the YM term, is indeed complex. This can be understood better if we realize that it is not just the value of κ that determines the appearance of regular islands, but also the available phase space as well.

The final picture which emerges bears out the fact that in a complex dynamical system with a large phase space (in contrast to the wide class of Hamiltonian systems with two degrees of freedom) curious interplay between different coupling constants and the rich structure of phase space itself can lead to novel results—some of them quite counter-intuitive and surprising. A detailed examination of dynamical systems which emerge from field theoretic systems is thus vital for an understanding of Hamiltonian systems with a large number of degrees of freedom. In turn, it is also an important primer in the understanding of non-Abelian field theories themselves.

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